1. **In this question you should show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Given

$$\frac{9^{x-1}}{3^{y+2}} = 81$$

express y in terms of x, writing your answer in simplest form.

(3)

$$\frac{9^{x-1}}{3^{y+2}} = 81 \longrightarrow \frac{3^{2x-2}}{3^{y+2}} = 3^4 \quad 1$$

$$= 3^{2x-2-(y+2)} = 3^4$$

We compare powers :

$$= 2x-2-y-2 = 4 \quad 1$$

$$y = 2x-8 \quad 1$$

2. The owners of a nature reserve decided to increase the area of the reserve covered by trees.

Tree planting started on 1st January 2005.

The area of the nature reserve covered by trees, A km^2, is modelled by the equation

$$A = 80 - 45e^{ct}$$

where c is a constant and t is the number of years after 1st January 2005.

Using the model,

(a) find the area of the nature reserve that was covered by trees just before tree planting started.

(1)

On 1st January 2019 an area of 60 km^2 of the nature reserve was covered by trees.

(b) Use this information to find a complete equation for the model, giving your value of c to 3 significant figures.

(4)

On 1st January 2020, the owners of the nature reserve announced a long-term plan to have 100 km^2 of the nature reserve covered by trees.

(c) State a reason why the model is not appropriate for this plan.

(1)

Before tree is planted, $t = 0$

a) When $t = 0$, $e^{ct} = 1$

$A = 80 - 45 \times 1 = 35$ km^2 (1)

$2005 \longrightarrow 2019 = 14$ years

b) $t = 14$ years. So,

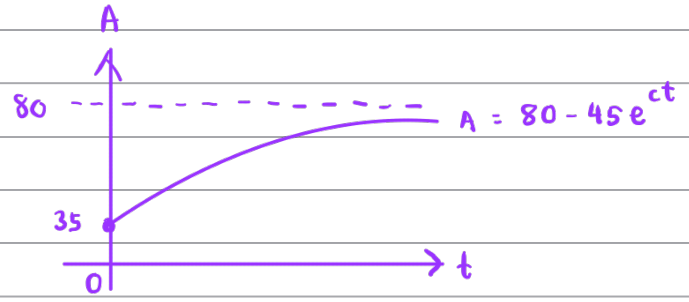
$A = 80 - 45e^{14c} = 60$ (1)

$\Rightarrow \quad 45e^{14c} = 20$ (1)

$\Rightarrow \quad c = \frac{1}{14} \ln\left(\frac{20}{45}\right)$

$c = -0.0579235\ldots$ (1)

$A = 80 - 45e^{-0.0579t}$ (1) (c to 3 sig. fig.)



c) The maximum area found by the model is 80 km^2, as $t \to \infty$. (1)

3. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The air pressure, P kg/cm^2, inside a car tyre, t minutes from the instant when the tyre developed a puncture is given by the equation

$$P = k + 1.4e^{-0.5t} \qquad t \in \mathbb{R} \qquad t \geqslant 0$$

where k is a constant.

Given that the initial air pressure inside the tyre was 2.2 kg/cm^2

(a) state the value of k.

(1)

From the instant when the tyre developed the puncture,

(b) find the time taken for the air pressure to fall to 1 kg/cm^2
Give your answer in minutes to one decimal place.

(3)

(c) Find the rate at which the air pressure in the tyre is decreasing exactly 2 minutes from the instant when the tyre developed the puncture.
Give your answer in kg/cm^2 per minute to 3 significant figures.

(2)

a) $t = 0,\ p = 2.2 \quad : \quad 2.2 = k + 1.4e^{0}$

$2.2 = k + 1.4$

$\therefore k = 0.8$ (1)

$\therefore p = 0.8 + 1.4e^{-0.5t}$

b) $p = 1 \quad : \quad 1 = 0.8 + 1.4e^{-0.5t}$

$1.4e^{-0.5t} = 0.2$ (1)

$e^{-0.5t} = \frac{1}{7}$

$-0.5t = \ln\left(\frac{1}{7}\right)$ (1)

$t = -2\ln\left(\frac{1}{7}\right)$

$\therefore t = 3.9$ minutes (1 d.p.) (1)

c) $\frac{dP}{dt} = -0.7e^{-0.5t}$ 1

when $t = 2$: $\frac{dP}{dt} = -0.7e^{-0.5(2)}$ 1 $= -0.2575...$

$\therefore$ decreasing at a rate of $0.258\ kg/cm^2$ (3 s.f.)
1

4. A scientist is studying the growth of two different populations of bacteria.

The number of bacteria, N, in the **first** population is modelled by the equation

$$N = Ae^{kt} \qquad t \geqslant 0$$

where A and k are positive constants and t is the time in hours from the start of the study.

Given that

- there were 1000 bacteria in this population at the start of the study
- it took exactly 5 hours from the start of the study for this population to double

(a) find a complete equation for the model.

(4)

(b) Hence find the rate of increase in the number of bacteria in this population exactly 8 hours from the start of the study. Give your answer to 2 significant figures.

(2)

The number of bacteria, M, in the **second** population is modelled by the equation

$$M = 500e^{1.4kt} \qquad t \geqslant 0$$

where k has the value found in part (a) and t is the time in hours from the start of the study.

Given that T hours after the start of the study, the number of bacteria in the two different populations was the same,

(c) find the value of T.

(3)

a) $N = Ae^{kt}$

given: at $t=0$, $N=1{,}000$
at $t=5$, $N=2{,}000$

sub in $t=0$, $N=1{,}000$ to find A:

$1000 = Ae^0 \qquad e^0 = 1$
$A = 1000$ (1)

sub in $t=5$, $N=2{,}000$ to find k

$2{,}000 = 1000e^{5k}$
$e^{5k} = 2$ (1)
$5k = \ln 2$
$k = \frac{1}{5}\ln 2$ (1)

$A = 1000$
$k = \frac{1}{5}\ln 2$

so $N = 1{,}000e^{\frac{t}{5}\ln 2}$ (1)

b) $N = 1000e^{\frac{t}{5}\ln 2}$

differentiate N to find rate of increase:

$$\frac{dN}{dt} = \left(\frac{1}{5}\ln 2\right) \times 1000e^{\frac{t}{5}\ln 2}$$

$$\frac{dN}{dt} = 200\ln 2 \times e^{\frac{t}{5}\ln 2}$$

$$\left.\frac{dN}{dt}\right|_{t=8} = 200\ln 2 \times e^{\frac{8}{5}\ln 2} \quad (1) \quad = 420.245...$$

$$= 420 \text{ (2sf)} \quad (1)$$

c) $M = 500e^{1.4kt}$ $\quad N = 1000e^{kt}$ $\quad \left(k = \frac{1}{5}\ln 2\right)$

Populations same when $t = T$

$$500e^{1.4kT} = 1000e^{kT} \quad (1)$$

$$\frac{e^{1.4kT}}{e^{kT}} = \frac{1000}{500}$$

$$e^{0.4kT} = 2$$

$$0.4kT = \ln 2$$

(1)

$$T = \frac{\ln 2}{0.4k} = \frac{25}{2} = 12.5 \text{ hours} \quad (1)$$

5. A scientist is studying the number of bees and the number of wasps on an island.

The number of bees, measured in thousands, N_b, is modelled by the equation

$$N_b = 45 + 220\,e^{0.05t}$$

where t is the number of years from the start of the study.

According to the model,

(a) find the number of bees at the start of the study,

(1)

(b) show that, exactly 10 years after the start of the study, the number of bees was increasing at a **rate** of approximately 18 thousand per year.

(3)

The number of wasps, measured in thousands, N_w, is modelled by the equation

$$N_w = 10 + 800\,e^{-0.05t}$$

where t is the number of years from the start of the study.

When $t = T$, according to the models, there are an equal number of bees and wasps.

(c) Find the value of T to 2 decimal places.

(4)

(a) when $t = 0$:

$$N_b = 45 + 220e^{0.05\times 0}$$
$$= 45 + 220e^{0} \quad \leftarrow e^{0} = 1$$
$$= 45 + 220$$
$$= 265$$

265 thousand (1)

(b) $\frac{dN_b}{dt} = 0.05 \times 220 \times e^{0.05t}$ ← differentiate w.r.t time to get rate of change.

$= 11e^{0.05t}$ (1) ← $\frac{d}{dx} e^{ax} = ae^{ax}$

when $t = 10$.

$$\frac{dN_b}{dt} = 11e^{0.05\times 10} \quad (1)$$
$$= 18.135...$$

which is approximately 18 thousand bees per year (1)

(c) when $t = T$, $N_b = N_w$:

$$45 + 220e^{0.05t} = 10 + 800e^{-0.05t} \quad (1)$$

$-10,\ -800e^{-0.05t}$

$$220e^{0.05t} + 35 - 800e^{-0.05t} = 0$$

$\times e^{0.05t}$

$$(1)\ 220(e^{0.05t})^2 + 35e^{0.05t} - 800 = 0$$

Do this to remove the $e^{-0.05t}$ term.

$e^{0.05t} \times e^{-0.05t} = e^0 = 1$

This is a quadratic $220x^2 + 35x - 800 = 0$ with $x = e^{0.05t}$. Solve with calculator.

$$e^{0.05t} = 1.829,\ -1.988$$

ignore negative result because e^n cannot be negative

$$0.05t = \ln(1.829) \quad (1)$$

$$t = 12.08 \quad (2dp)$$

$\therefore T = 12.08$ years (1)

6. Coffee is poured into a cup.

The temperature of the coffee, H °C, t minutes after being poured into the cup is modelled by the equation

$$H = Ae^{-Bt} + 30$$

where A and B are constants.

Initially, the temperature of the coffee was 85 °C.

(a) State the value of A.

(1)

$\frac{dH}{dt}$ is negative

Initially, the coffee was cooling at a rate of 7.5 °C per minute.

(b) Find a complete equation linking H and t, giving the value of B to 3 decimal places.

(3)

a) when $t=0$, $H=85$

$85 = Ae^{0} + 30$

$A = 55$ (1)

b) when $t=0$, $\frac{dH}{dt} = -7.5$

$H = 55e^{-Bt} + 30$

$\frac{dH}{dt} = -55Be^{-Bt}$ (1)

$-7.5 = -55Be^{0}$

$B = \frac{3}{22} = 0.136$ (3sf) (1)

$H = 55e^{-0.136t} + 30$ (1)